EE-305

[FUNDAMENTALS OF CONTROL SYSTEM]

Question-1 Derive ~~Derive~~ the expression of time response of second order system for underdamped case when the input is unit step.

Answer The standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

where ω_n = Undamped natural frequency, rad/sec.
ζ = Damping Ratio

For underdamped system, $0<\zeta<1$ and roots of the characteristic equation are complex conjugate.

The roots of the characteristic equation are

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

since $\zeta<0$, $\zeta^2<1$

$$\therefore s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The damped frequency of oscillation

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain $C(s) = R(s)\dfrac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}$

For unit step input $r(t)=1$ and $R(s)=\frac{1}{s}$

$$\therefore C(s) = \frac{1}{s} * \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

By partial fraction expansion,

$$C(s) = \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\Rightarrow \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A(s^2 + 2\zeta\omega_n s + \omega_n^2) + (Bs + C)s}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

put $s = 0$ $\quad A = \frac{\omega_n^2}{\omega_n^2} = 1$

$$\boxed{A = 1}$$

To solve for B and C

$$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{s} + \frac{Bs + C}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

On cross multiplication and equating the coefficient

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get $0 = 1 + B$

$$\therefore \boxed{B = -1}$$

Equating coefficient of s we get $0 = 2\zeta\omega_n + C$

$$\therefore \boxed{C = -2\zeta\omega_n}$$

$$C(s) = \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Adding and substracting $\zeta^2\omega_n^2$ to the denominator of second term in the above equation

$$C(s) = \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + 2\zeta\omega_n)}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

$$\therefore \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

multiplying and dividing by ω_d in the third term of above equation

$$C(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\omega_n\zeta)^2+\omega_d^2}$$

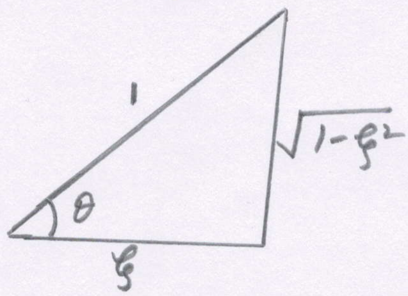
on taking inverse Laplace transform

$$C(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\omega_d t\cdot\zeta + \sqrt{1-\zeta^2}\cos\omega_d t\right]$$

on construction right angle. triangle with ζ and $\sqrt{1-\zeta^2}$, we get



$$\sin\theta = \sqrt{1-\zeta^2}$$

$$\cos\theta = \zeta$$

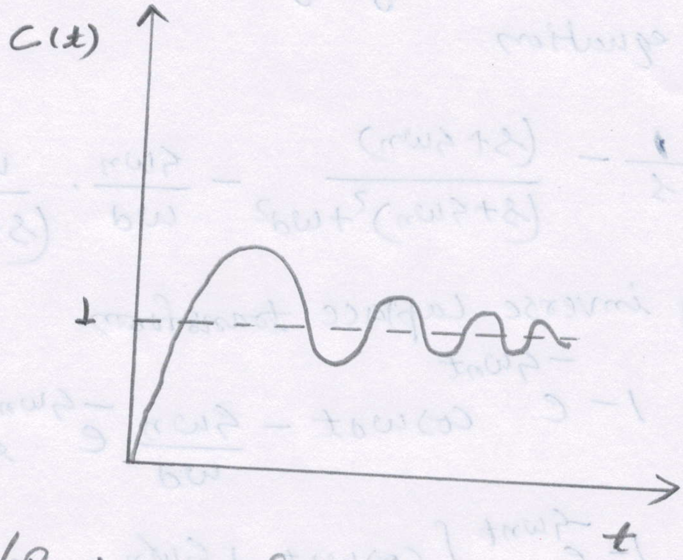
$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\omega_d t\cos\theta + \cos\omega_d t\sin\theta\right]$$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

where $\theta = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$

The response of under damped second order system oscillates before settling to a final value



(Response of under damped second order system for unit step input)